Calculus

Part I (70%) - Fill in the blanks

7% each, NO partial credits.

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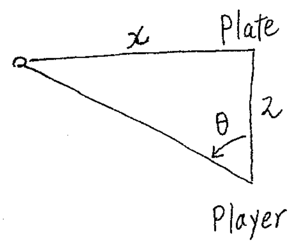
(1) Consider the function $f(x) = \sin(2x) - 2\sin x$, where $x \in [-\pi, \pi]$. $f(x)$ has 3 points of inflection and their x coordinates are given by ______.

(2) The directional derivative of $f(x,y,z) = x^2 + yz$ at $(1,-3,2)$ in the direction of the path $\mathbf{r}(t) = t^2\mathbf{i} + 3t\mathbf{j} + (1-t^3)\mathbf{k}$ is ______.

(3) $\displaystyle\lim_{x\to\infty} \frac{1}{x}\int_0^x \sin\left(\frac{1}{1+t}\right) dt =$ ______. (Answer "None" if the limit does not exist.)

(4) A baseball player stands 2 feet from home plate and watches a pitch fly by. In the diagram, x is the distance from the ball to home plate and θ is the angle indicating the direction of the player's gaze. The instantaneous rate of change θ' of the angle θ at which his eyes must move to watch a fastball with $x' = -130$ ft/s (feet per second) as it crosses home plate at $x = 0$ is ______. Given the fact that humans can maintain focus only when $\theta' \leq 3$, the fastest pitch that you could actually watch cross home plate is ______.



(5) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(2x)}{(\sin x)^{4/3}} dx =$ ______, $\displaystyle\int_0^1 x\ln(x+3)dx =$ ______.

(6) The value of the infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$ is ______. (Answer "Diverges" if the series is not convergent.)

(7) A rocket is launched with a constant thrust corresponding to an acceleration of u ft/s^2. Ignoring air resistance, the rocket's height after t seconds is given by $f(t,u) = \frac{1}{2}(u-32)t^2$ feet. Fuel usage for t seconds is proportional to $u^2 t$, so the limited fuel

capacity of the rocket can be expressed by an equation of the form $u^2 t = 10,000$. The value of u that maximizes the height that the rocket reaches is ______.

(8) Let Ω be the parallelogram bounded by the $x+y=0$, $x+y=1$, $x-y=0$ and $x-y=2$. Then $\iint_{\Omega}(x^2-y^2)dxdy =$ ______.

PART II (30%) - COMPUTATIONAL PROBLEMS

Show all your work. NO CREDITS if only present answers.

(1) Let f and g be two functions defined and having finite third order derivatives $f'''(x)$ and $g'''(x)$ for all $x \in \mathbb{R}$. If $f(x)g(x) = \pi$, show that the following relations hold at those points where the denominators are nonzero.

(a) (7%) $\dfrac{f'(x)}{f(x)} + \dfrac{g'(x)}{g(x)} = 0.$

(b) (8%) $\dfrac{f''(x)}{f'(x)} - 2\dfrac{f'(x)}{f(x)} - \dfrac{g''(x)}{g'(x)} = 0.$

(2) A function $F(x,y,z)$ is called homogeneous of degree p if, for all values of the parameter λ, we have the identity

$$F(\lambda x, \lambda y, \lambda z) = \lambda^p F(x,y,z).$$

(a) (7%) Is the function $G(x,y,z) = x^4 y^2 z \tan^{-1}(z/x)$ homogeneous? If yes, what is the degree? Show all your works.

(b) (8%) Prove that if $F(x,y,z)$ is differentiable and homogeneous of degree p, then $F(x,y,z)$ satisfies

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} + z\frac{\partial F}{\partial z} = pF.$$